If $\xi^i(t)$ in all of Eqs. (10) is the solution of (5) subject to (6) and (7), all values in the $n - r$ Eqs. (10) are known except $\xi^i(t_0)$ for $i = r + 1, \ldots, n$, and hence these values can be found.

The same result can be obtained with very little more labor without the use of the adjoint equations. Let $\xi_k{}^i(t)$ for $k = r + 1, \ldots, n$ be $n - r$ solutions of (5) subject to the initial conditions

$$\xi_k{}^i(t_0) = 0 \qquad (i = 1, \ldots, r)$$
$$\xi_k{}^i(t_0) = \delta_k{}^i \qquad (i = r + 1, \ldots, n)$$

These solutions determine the values $\xi_k{}^i(t_1)$. If $\mu^{r+1}, \ldots, \mu^n$ are any $n - r$ constants

$$\xi^i(t) = \mu^{r+1}\xi_{r+1}{}^i(t) + \ldots + \mu^n\xi_n{}^i(t) \tag{11}$$

is a solution of (5) satisfying (6). If this solution is also to satisfy (7)

$$\beta^i = \mu^{r+1}\xi_{r+1}{}^i(t_1) + \ldots + \mu^n\xi_n{}^i(t_1) \qquad (i = r + 1, \ldots, n)$$

These $n - r$ equations can be solved for $\mu^{r+1}, \ldots, \mu^n$, the solution values substituted in (11), and then putting $t = t_0$ for $i = r + 1, \ldots, n$ yields the required values of $\xi^i(t_0)$. Thus, the steps in this method are exactly parallel to those in the adjoint method except that here one additional computation (equivalent to a matrix $\times$ vector multiplication) is necessary after the simultaneous linear equations have been solved. It hardly seems worth the effort of introducing the adjoint system to avoid this simple step, especially in an exposition of principles.

The first use of the adjoint system in problems having a superficial resemblance to that considered here was by Bliss[3, 4] in his work in ballistics during WW I, but in these applications it serves a much more useful purpose. A simple example of this kind is that in which the $\xi^i(t)$ are variations from a normal trajectory due to abnormal initial conditions and an expression for the final value of just one of the $\xi^i(t)$ is required, in terms of arbitrary initial variations of all of the $\xi^i(t)$. The coefficients in this expression can be obtained with only one integration of the adjoint equations, whereas if the same expression were to be obtained by integrating the equations of variation, n integrations would be required.

References

[1] Jurovics, S. A. and McIntyre, J. E., "The adjoint method and its application to trajectory optimization," ARS J. **32**, 1354–1358 (1962).

[2] Melbourne, W. G., "Space flight optimization," NASA SP-15, 29-41 (1962).

[3] Bliss, G. A., *Mathematics for Exterior Ballistics* (John Wiley and Sons, Inc., New York, 1944), Chap. V.

[4] McShane, E. J., Kelley, J. L., and Reno, F. V., *Exterior Ballistics* (The University of Denver Press, Denver, Colo., 1953), Chap. IX.

Optimum Planar Circular Orbits Transfer

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Introduction

A RECENT note[1] points out that the solution obtained by Jurovics and McIntyre[2] for the minimum time transfer of a constant thrust acceleration vehicle between two coplanar earth circular orbits is in error. It is apparent from the lengthy footnote in Jurovics and McIntyre's paper that they had considerable difficulties in meeting boundary conditions. Their method of solving the variational boundary value problem has been developed independently by this author[3–5] and successfully applied to the problem at hand as well as many other problems. Convergence has been excellent in all problems solved. These results demonstrate that Jurovics and McIntyre's solution is erroneous (the optimal control is continuous).

It is the purpose of this note to exhibit a continuous optimal control for the problem under discussion, which everywhere satisfies the Legendre-Clebsch necessary condition, and to show that a discontinuous control is nonoptimal. The results obtained by Jurovics and McIntyre are not inherent to the method of solution used, which is demonstrably a very good one.

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Discussion

The problem was formulated in terms of the (v, γ, h) state, where v is the total velocity, γ the local flight path angle, and h the altitude. Control is embodied in α, the angle between the velocity and thrust vectors. A minimum time transfer from a 300- to a 1000-statute mile circular earth orbit was obtained for a vehicle with a constant ratio of (thrust acceleration)/(initial gravitational acceleration) equal to 1.7343. Pertinent constants are GM (universal gravitational constant $\times$ earth mass) $= 1.408142 \times 10^{16}$ ft^3/sec^2, and R (earth radius) $= 2.09029 \times 10^7$ ft. This transfer corresponds to a ratio of (radial transfer distance)/(initial orbit radius) equal to 0.164377—very nearly the problem considered by Jurovics and McIntyre[2] and Greenley.[1]

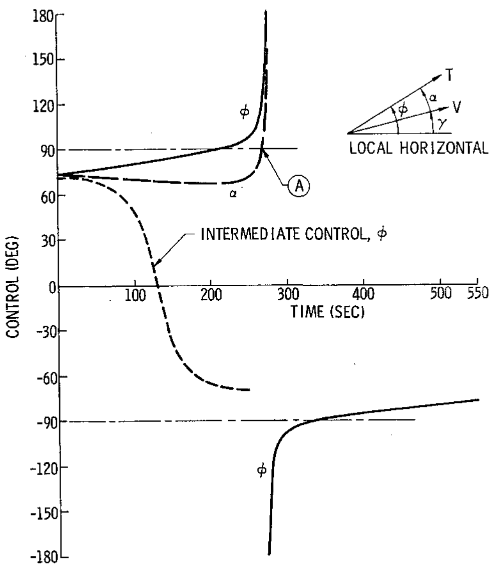

Fig. 1 Optimal control.

The minimum time control obtained is plotted in Fig. 1 as ϕ, the angle between the local horizontal and the thrust direction, which is the control considered by Jurovics and McIntyre. A ratio of (minimum transfer time)/(time per rad in initial orbit) equal to 0.612160 is obtained. The boundary conditions imposed on the variational boundary value problem were met to seven significant figures.

The control α for this problem may be written in terms of the Lagrange multipliers

$$\alpha = \tan^{-1}(\mu_\gamma / v\mu_v) \tag{1}$$

where μ_v is the multiplier associated with the velocity differential constraint, and μ_γ is the multiplier associated with the path angle differential constraint. It is observed that α is multivalued. The appropriate α may be chosen with the aid of the Legendre-Clebsch necessary condition. This condition may be written (for a minimum)

$$\mu_v \cos\alpha + \mu_\gamma(\sin\alpha/v) \geq 0 \qquad (2)$$

In solving the variational boundary value problem, it must be ascertained that inequality (2) is satisfied at each point of the solution.

An intermediate control (i.e., not satisfying the desired boundary conditions) is shown in Fig. 1. As the iterations proceed, the control is reshaped until at some point (A in Fig. 1), α† assumes the value of 90°. At this point $\mu_v = 0$ and $\mu_\gamma > 0$ so that the Legendre-Clebsch condition reduces to

$$C \sin\alpha \geq 0 \qquad C > 0 \qquad (3)$$

It is clear that a discontinuity in α from +90° to −90° is not permitted since $\alpha = -90°$ violates condition (3). The arc beyond such a discontinuity is a nonminimal one. A continuous control, on the other hand, does satisfy condition (3). It may be noted that a discontinuity can be allowed only if the equality in Eq. (2) holds. The control given in Fig. 1 everywhere satisfies the Legendre-Clebsch condition.

In closing, it may be emphasized that the origin of the boundary value problem cannot be forgotten in seeking its solution. The Euler-Lagrange equations are not the only conditions that a minimizing arc must satisfy.

References

[1] Greenley, R. R., "Comments on 'The adjoint method and its applications to trajectory optimization,'" AIAA J. 1, 1463 (1963).

[2] Jurovics, S. A. and McIntyre, J. E., "The adjoint method and its application to trajectory optimization," ARS J. 32, 1354–1358 (1962).

[3] Jazwinski, A. H., "Integration of two-point boundary value problems with one unknown boundary," General Dynamics/Astronautics A-PD-113 (May 17, 1962).

[4] Jazwinski, A. H., "New approach to variational problems in trajectory optimization," Martin Co., Space Systems Div., ER 12700 (November 1962).

[5] Jazwinski, A. H., "Optimum trajectories by application of Green's theorem to the variational boundary value problem," Martin Co., Space Systems Div., ER 12943 (March 1963).

† Only a portion of the α-optimal control is shown in Fig. 1.

Comment on "Wind-Tunnel Interference for Wing-Body Combination"

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I HAVE studied with interest Gorgui's analysis of the wind-tunnel interference for a wing-body combination.[1] Although I have obtained the same result for a circular tunnel, using the method of images, I believe that the results are misleading and the conclusion inaccurate.

It is usual to allow for the mean interference by means of a correction to the angle of attack, and one is then interested in the spanwise variation of interference which has not been accounted for by this correction.

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Now, the upwash induced by a change in angle of attack is not uniform over the span of a wing-body combination. Indeed, by considering uniform flow past a circular cylinder, it can be shown that the upwash is doubled in the vicinity of the wing-body junction. No allowance has been made in Gorgui's analysis for the change in body angle of attack, and this explains the variation in δ near the wing-body junction.

It appears that the curves for ($r/s = 0$) are valid for determining the correction to angle of attack and the residual interference for a wing-body combination. There is not, as suggested, any tendency towards a root stall, apart from that experienced at the corrected incidence in free flight. This conclusion is important, in view of the importance attached to stall development work in wind tunnels.

Reference

[1] Gorgui, M. A., "Wind-tunnel interference for wing-body combination," J. Aerospace Sci. 28, 823–825 (1961).

Correlation of the Critical Pressure of Conical Shells with That of Equivalent Cylindrical Shells

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IN Ref. 1, Seide showed that the critical pressures for isotropic conical shells under hydrostatic pressure can be correlated to those of equivalent cylindrical shells. The correlation yielded an approximate curve for the ratio of the critical pressure of conical shells to that of their equivalent cylindrical shells (Fig. 2 of Ref. 1). A very similar curve was obtained in Ref. 2 for conventional simple supports (which differ slightly from Seide's boundary conditions).

However, in both papers the calculations did not include large cone angles. Recent computations indicate that for larger cone angles the single curve should be replaced by a family of curves. Reappraisal of Fig. 2 of Ref. 1 brings out this cone angle dependence for 60°, as can be seen in Fig. 1, where that figure is reproduced with emphasis on the 60° points. (The remainder were for 10°, 20°, 30°, and 45°.) It is apparent that a better fitting correlation curve can be obtained if only cone angles up to 45° are included (or even better if only up to 30°) and the 60° points are joined by a similar curve.

Further computations for conventional simple supports by the method of Ref. 2 yields a family of correlation curves given in Fig. 2. The curves show the ratio of the critical pressure p of a conical shell to that of an equivalent cylindrical shell $\bar{p}$ vs the taper ratio ($1-R_1/R_2$). The equivalent cylindrical shell is defined as one having the same thickness as the conical shell, but whose radius is the mean radius of curvature of the cone and whose length is that of its slant length. As may be seen, the 60° curve deviates only slightly, whereas the 75° and 85° curves are noticeably lower. The actual percentage reduction in the ($p/\bar{p}$) ratio is only of the order of a few percent (up to about 6–7% for a large taper ratio and a cone angle of 85°), but since it is unconservative it is significant.

The computations brought out another nonconservative secondary effect. Seide's correlation curve[1] and that of Ref.

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